

Total coloring of 1-toroidal graphs with some restrictions on triangles

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Abstract

A graph is *1-toroidal* if it can be drawn on torus such that every edge cross at most one other edge. Total Coloring Conjecture (TCC) says that every graph with maximum degree Δ admits a total- $(\Delta + 2)$ -coloring. In this paper, we prove TCC holds for the 1-toroidal graphs with maximum degree at least 11 and some restrictions on the triangles. Consequently, if G is a diamond-free 1-toroidal graph with maximum degree at most Δ , where $\Delta \geq 11$, and every subgraph K_4 has a vertex of degree at most four, then G admits a total- $(\Delta + 2)$ -coloring.

1 Introduction

All graphs considered are finite, simple and undirected unless otherwise stated. Let G be a graph with vertex set V and edge set E . The *neighborhood* of a vertex v in a graph G , denoted by $N_G(v)$, is the set of all the vertices adjacent to the vertex v , i.e., $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. The *degree* of a vertex v in G , denoted by $\deg_G(v)$, is the number of edges of G incident with v . We denote the minimum and maximum degrees of vertices of G by $\delta(G)$ and $\Delta(G)$, respectively. The *diamond graph* K_4^- is the graph K_4 minus an edge. A graph property \mathcal{P} is *deletion-closed* if \mathcal{P} is closed under removal of edges.

A *total coloring* of a graph G is an assignment of colors to the vertices and the edges of G such that every pair of adjacent/incident elements receive distinct colors. The *total chromatic number* of a graph G , denoted by $\chi''(G)$, is the minimum number of colors needed in a total coloring of G . It is obviously that the total chromatic number has a trivial lower bound $\Delta(G) + 1$. For the upper bound, Behzad raised the following well-known Total Coloring Conjecture (TCC):

Total Coloring Conjecture (Behzad [1]). Every graph with maximum degree Δ admits a total- $(\Delta + 2)$ -coloring.

This conjecture was verified by Rosenfeld [9] and Vijayaditya [11] independently and also by Yap [12] for $\Delta = 3$. It was confirmed by Kostochka [6, 7] for $\Delta = 4, 5$, in fact the proof holds for multigraphs. For planar graphs, the conjecture was verified by Borodin [3] for $\Delta \geq 9$, and by Sanders and Zhao [10] for $\Delta = 7$. So it suffices to consider the planar graphs with maximum degree 6.

TCC is equivalent to the following conjecture, so we prove [Conjecture 1](#) instead of TCC.

Conjecture 1. Every graph with maximum degree at most Δ admits a total- $(\Delta + 2)$ -coloring.

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The minimal counterexample to [Conjecture 1](#) has the following properties:

- (P1) Let x and y be two adjacent vertices in G . If $\deg_G(x) \leq \lfloor \frac{\Delta+1}{2} \rfloor$, then $\deg_G(y) \geq \Delta + 3 - \deg_G(x)$.
- (P2) The graph G is 2-connected and $\delta(G) \geq 3$.
- (P3) Let u, v be two adjacent vertices of G and $\deg_G(v) \leq \lfloor \frac{\Delta+1}{2} \rfloor$. If $\deg_G(u) + \deg_G(v) = \Delta + 3$, then the edge uv is not contained in a triangle of G .

Proof. Suppose that uv is contained in a triangle uvw . By the minimality of G , the graph $G - uv$ has a total- $(\Delta + 2)$ -coloring f . Erase the color of v , then a color which is missed at u must appear at v and a color which is missed at v must appear at u , otherwise, assign a color which is missed at u and v to uv and give a suitable color to v , it yields a total- $(\Delta + 2)$ -coloring of G , which is a contradiction. Therefore, there is no common colors at u and v and $\{1, \dots, \Delta + 2\}$ is the disjoint union of the colors appears at u and v . From the coloring f , erase the color of v and uv , and assign color $f(uv)$ to uv , obtained a total- $(\Delta + 2)$ -coloring of $G - uv$ except v . Similarly, we can prove that $\{1, \dots, \Delta + 2\}$ is the disjoint union of colors appears at w and v . Therefore, in the coloring of f , there is a color m missed at u and w , and hence recolor uw with m , assign $f(uv)$ to uv , and give a suitable color to v , which yields a total- $(\Delta + 2)$ -coloring of G , which derive a contradiction. \square

- (P4) If v is a vertex of degree three, then $N_G(v)$ is an independent set [[13](#), Lemma 2].
- (P5) If v is a vertex of degree four, then no edge at v is contained in two triangles [[13](#), Lemma 3].
- (P6) There is no 3-alternating cycle, that is, a cycle $v_0v_1v_2 \dots v_{2k-1}v_0$ such that $\deg_G(v_{2i}) = 3$ for $i = 0, 1, 2, \dots, k-1$.

A graph is 1-embeddable in a surface S if it can be drawn on S such that every edge cross at most one other edge. In particular, a graph is 1-toroidal if it can be drawn on torus such that every edge cross at most one other edge; a graph is 1-planar if it can be drawn on the plane such that every edge cross at most one other edge. The concept of 1-planar graph was introduced by Ringel [[8](#)] in 1965, while he simultaneously color the vertices and faces of a plane graph such that any pair of adjacent/incident elements receive different colors. Ringel [[8](#)] proved that 1-planar graphs are 7-colorable, and conjectured that they are 6-colorable, this conjecture was proved to be true by Borodin [[2](#), [4](#)].

Obviously, planar graphs are 1-planar graphs, and 1-planar graphs is an extension of planar graphs in some sense. Zhang et al. [[13](#)] proved the TCC holds for 1-planar graphs with maximum degree at least 13. For the other various coloring of 1-planar graphs, see [[5](#), [14–16](#)]. In this paper, we consider the total coloring of 1-toroidal graphs, and prove that the TCC holds for the 1-toroidal graphs with some restriction on the triangles.

Let G be a graph having been drawn on a surface, if we treat all the crossing points as vertices, and then obtained an embedded graph G^\dagger , and call it the associated graph of G , call the vertices of G true vertices and the crossing points crossing vertices.

2 Theorem

A graph G has property \mathcal{P} , if G satisfies the two conditions:

- (1) every subgraph K_4 has a vertex of degree at most four;
- (2) every induced subgraph K_4^- has a vertex on the common edge of degree at most five or has another vertex of degree three.

Theorem 2.1. Let G be a 1-toroidal graph with maximum degree at most Δ , where $\Delta \geq 11$. If G satisfies property \mathcal{P} , then G admits a total- $(\Delta + 2)$ -coloring.

Consequently,

Corollary 1. If G is a diamond-free 1-toroidal graph with maximum degree at most Δ , where $\Delta \geq 11$, and every subgraph K_4 has a vertex of degree at most four, then G admits a total- $(\Delta + 2)$ -coloring.

Corollary 2. Let G be a 1-toroidal graph with maximum degree at most Δ , where $\Delta \geq 11$. If G has no adjacent triangles, then G admits a total- $(\Delta + 2)$ -coloring.

We prove the [Theorem 2.1](#) by contradiction. Let G be a minimal counterexample to the theorem and it has been 2-cell embedded on the plane/torus (that is, every face is homeomorphic to an open disc). The property \mathcal{P} is deletion-closed, then G is also a minimal counterexample to [Conjecture 1](#), and the properties (1)–(5) holds for G . Let G^\dagger be the associated graph of G . Hence G^\dagger is also 2-connected and every face boundary is a cycle of G^\dagger . A face f is called a *big face* if its size is at least four. If there exists a big face with two discontinuous true vertices of degree at most five, then add a line linking these two vertices in this face, and call this line a *new edge*. After recurrence adding new edges, we obtain an embedded graph G^* . By the construction, every face boundary of G^* is also a cycle. Note that G^* maybe have multiple edges, but if there exists two multiple edges e_1 and e_2 , then they are all new edges, since G and G^\dagger are simple graphs. Also, the crossing vertices are independent in G^* .

A vertex in G^* is called a (k, l) -vertex, if it is of degree k in G and of degree l in G^* . A vertex v is called *big* if it is a $(3, 5)$ -vertex or $\deg_{G^*}(v) \geq 6$; otherwise, it is called a *small* vertex (including the crossing vertices).

By Euler's formula, we have

$$\sum_{v \in V(G^*)} (\deg_{G^*}(v) - 6) + \sum_{f \in F(G^*)} (2 \deg_{G^*}(f) - 6) \leq 0$$

We will use discharging method to complete the proof. The initial charge of every vertex v is $\deg_{G^*}(v) - 6$, and the initial charge of every face f is $2 \deg_{G^*}(f) - 6$. Then the sum of charge of vertices and faces is at most zero by the Euler formula. We then transfer some charge from the big faces and some big vertices to small vertices, such that the final charge of every small vertex becomes nonnegative and the final charge of every big vertex remains nonnegative, but there is at least one element's final charge is positive, and thus the sum of the final charge of vertices and faces is positive, which derive a contradiction.

Lemma 1. There is no four pairwise adjacent vertices in G .

Proof. Suppose that $\{v_1, v_2, v_3, v_4\}$ induce a K_4 in G , then one of the four vertices, say v_1 , has degree at most four by the hypothesis of the theorem. If $\deg_G(v_1) = 3$, then v_1 is contained in a triangle $v_1 v_2 v_3 v_1$, which contradicts (P4). If $\deg_G(v_1) = 4$, then the edge $v_1 v_3$ is contained in two adjacent triangles in G , which contradicts (P5). \square

Lemma 2. Let uvw be on a face boundary of a big face of G^* . If u is a true vertex of degree at most five and uw is not a new edge, then at least one of v and w is a big vertex in G^* .

Proof. If v is a true vertex, then $\deg_{G^*}(v) \geq \Delta - 2 \geq 9$. So we may assume that v is a crossing vertex. Hence, the vertex w is a true vertex, moreover, it is a big vertex in G^* ; otherwise, u and w should be linked in this face, a contradiction. \square

The Discharging Rules:

(R1) Every Δ -vertex which is adjacent to a $(3, *)$ -vertex of G^* sends $1/2$ to a special Δ -vertex v_0 , and then every $(3, *)$ -vertex of G^* receive 1 from v_0 ;

- (R2) Every big face donates its redundant charge equally to vertices whose charge is negative. In other words, every big face donates its redundant charge equally to small vertices.
- (R3) Some other discharging rules, see the figures (a)–(x); note that the dashed line denotes the two vertices are nonadjacent and the wavy line denotes the “new edge”; the solid dot denotes true vertex and the hollow dot denotes crossing vertex.

Lemma 3. Let uvw be on a face boundary of a big face f . Suppose that v is a true vertex of degree at most five and uv, vw are not new edges. If f is a 4-face, then v receive at least 1 from f , unless u, w are all crossing vertices and v receive $2/3$ from f . If f is a 5^+ -face, then v receive at least $4/3$ from it.

Proof. If f is a 4-face and u, w are all crossing vertices, then f donates charge 2 equally to three small vertices by Lemma 2 and R2, hence v will receive $2/3$ from f . If $f = uvww'$ is a 4-face and at least one of $\{u, w\}$, say u , is not a crossing vertex, then u is a true (big) vertex, and also at least one of $\{w, w'\}$ is a big vertex by Lemma 2, the vertex v will receive at least 1 from f .

Assume that f is a 5^+ -face and $u'uvw w'$ is on the face boundary of f . By Lemma 2, at least one vertex of $\{u', u\}$ (similarly, $\{w, w'\}$) is a big vertex, then f is incident with at least two big vertices. Hence, the vertex v will receive at least

$$\frac{2 \deg(f) - 6}{\deg(f) - 2} = 2 - \frac{2}{\deg(f) - 2} \geq 2 - 2/3 = 4/3.$$

□

From the discharging rules, the final charge of every face is nonnegative. So it suffices to consider the final charge of vertices in G^* . Let v be an arbitrary vertex of G^* , we will analyse the vertex v according to its degree case by case.

Let e_0, e_1, \dots, e_k be consecutive edges at a vertex v with degree at least $\Delta - 2$, and the other end of e_i is v_i for $i = 0, 1, \dots, k$. If both v_0 and v_k receive 0 from v through e_0 and e_k respectively, and v_i receive positive charge from v for every $i = 1, \dots, k - 1$, we call this local structure a *semi-fan with k faces* and the vertex v *center* of the semi-fan, call the edges e_i *fan ribs*, and e_{i-1} *precursor* of e_i and e_{i+1} *successor* of e_i . We show that the vertices receive charge from big vertices such that its final charge is nonnegative and in every semi-fan, the average charge sent out by the center is at most $2/5$, and then the final charge of $(\Delta - 2)^+$ vertices are positive.

From the discharging rules, we also have the following claim:

Claim 1. Let w be a crossing vertex with a small neighbor w_1 . If w is incident with a 3-face with face angle $w_1 w w_2$ and ww_1 is incident with one big face, then w_2 will not send charge to w .

Case 1. The vertex v is a $(3, 3)$ -vertex and v_1, v_2, v_3 are its neighbors.

If v is incident with three 3-faces, then either there are two adjacent crossing vertices or v is contained in a triangle of G , which is a contradiction. Hence, the vertex v is incident with at least one big face.

Subcase 1.1. Suppose that v is incident with three big faces. By Lemma 3, the vertex v receives at least $2/3$ from each of its incident faces, and hence its final charge is at least $3 - 6 + 1 + 3 \times 2/3 = 0$ by R1.

Subcase 1.2. Assume that v is incident with two big faces. If v is incident with a 5^+ -face f , then it receives at least $4/3$ from f and receives at least $2/3$ from the other big face by Lemma 3, and hence its final charge is at least $3 - 6 + 1 + 4/3 + 2/3 = 0$. So we may assume that v is incident with two 4-faces and one 3-face. Without loss of generality, assume that v is incident with a 3-face with face angle $v_1 v v_2$ and v_1 is a true vertex. Then v_2 is a crossing vertex since v is not contained in a triangle of G . If v_3 is a true vertex, then v receives at least 1 from each of its incident 4-faces by Lemma 3, the final charge of v is at least $3 - 6 + 1 + 2 \times 1 = 0$. So we may assume that v_3 is a crossing vertex, see Fig (a). By R1 and Lemma 3, the final charge of v is at least $3 - 6 + 1 + 1 + 2/3 + 1/3 = 0$. From Claim 1, we know that the vertex v_1 does not send any charge to v_2 , and also does not send any charge to v^* for v^* is a big vertex.

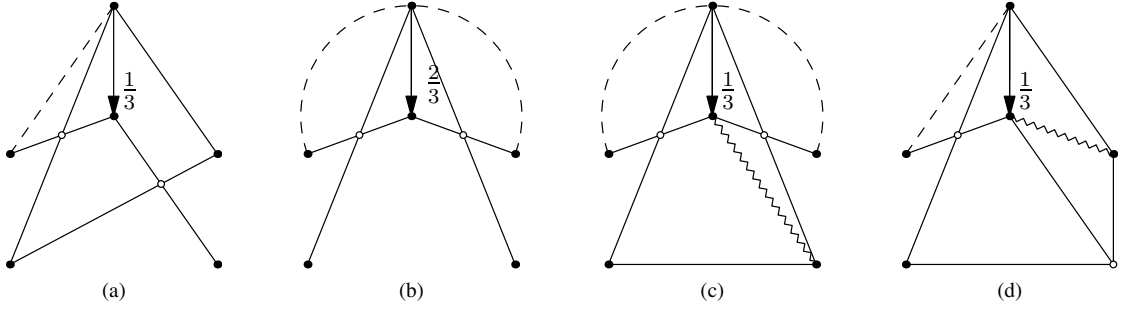


Fig. 1

Subcase 1.3. Assume that v is incident with only one big face f . Without loss of generality, assume that f has a face angle v_2vv_3 . If v_1 is a crossing vertex, then v_2 and v_3 are all true vertices, and thus v is contained in a triangle of G induced by v_2, v, v_3 , which contradicts (P4). So we may assume that v_1 is a true vertex and v_2, v_3 are all crossing vertices for the same reason. In fact, the big face f is a 5^+ -face, otherwise, there exists two multiple edges in G , which is a contradiction. By R1 and Lemma 3, the final charge of v is at least $3 - 6 + 1 + 4/3 + 2/3 = 0$, see Fig (b). From Claim 1, the vertex v_1 does not send any charge to v_2 and v_3 .

Let v_1, v_2, \dots, v_l be v 's neighbors in counterclockwise order, and f_i be the incident face with face angle v_ivv_{i-1} , where the subtraction of subscript is taken modulo l .

Case 2. The vertex v is a $(3, 4)$ -vertex, that is, v is a 4-vertex in G^* and it is incident with a new edge.

Subcase 2.1. If v is incident with at least two big faces, then its final charge is at least $4 - 6 + 1 + 2 \times 1/2 = 0$ by R1 and R2.

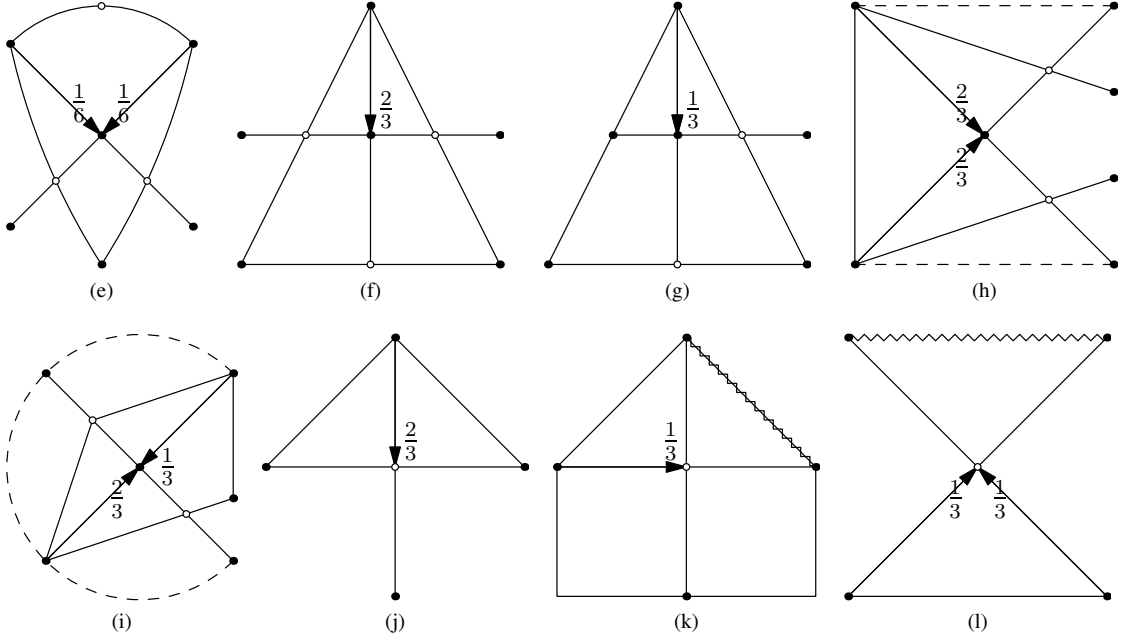
Subcase 2.2. Assume that v is incident with one big face f and the new edge at v is incident with f . By a similar argument as in subcase 1.3, we may assume that v_1 is a true vertex and v_2, v_4 are crossing vertices and f has the face angle v_2vv_3 . If f is a big face with degree at least five, then v receives at least 1 from f , and thus the final charge of v is at least $4 - 6 + 1 + 1 = 0$. So we may assume that f is a 4-face, see Fig (c). By R1, R2 and Lemma 2, the final charge of v is at least $4 - 6 + 1 + 2/3 + 1/3 = 0$. From Claim 1, the vertex v_1 does not send any charge to v_2 .

Suppose that v is incident with one big face f and the new edge at v is not incident with f . If v receives at least 1 from the big face f , then its final charge is at least $4 - 6 + 1 + 1 = 0$. So we may assume that v receives less than 1 from f . By Lemma 3, f is a 4-face and v is incident with two crossing vertices on f , see Fig (d). By R1 and Lemma 3, the final charge of v is at least $4 - 6 + 1 + 2/3 + 1/3 = 0$. Also, from Claim 1, the vertex v_1 does not send any charge to v_2 .

Subcase 2.3. Assume that v is incident with four 3-faces and vv_1 is the new edge at v . If v_3 is a crossing vertex, then v_2 and v_4 are all true vertices, but then v is contained in a triangle on three vertices v_2, v, v_4 . So we may assume that v_3 is a true vertex, and hence v_2 and v_4 are crossing vertices by (P4), but in this case there are two multiple edges of G with ends v_1 and v_3 , which is a contradiction. Therefore, it is impossible to have four 3-faces incident with v .

Case 3. The vertex v is a $(4, 4)$ -vertex.

Subcase 3.1. Assume that v is incident with at least three big faces. By Lemma 3, the vertex v receives at least $2/3$ from each of its incident big faces, then the final charge of v is at least $4 - 6 + 3 \times 2/3 = 0$.



Subcase 3.2. Assume that v is incident with two big faces. If v receives at least 1 from each of its incident big faces, then its final charge is at least $4 - 6 + 2 \times 1 = 0$. By Lemma 3, we may assume that v receives $2/3$ from one of its incident 4-face with face angle v_3vv_4 . Moreover, the vertices v_3 and v_4 are all crossing vertices. If v receives at least $4/3$ from the other big face, then its final charge is at least $4 - 6 + 2/3 + 4/3 = 0$. So we may assume that the other big face sends at most 1 to v and it is also a 4-face.

Assume that the two 4-faces are nonadjacent, then v_1 and v_2 are all true vertices, see Fig (e). By R2, the vertex v receives at least 1 from f_2 . Hence, the final charge of v is at least $4 - 6 + 1 + 2/3 + 2 \times 1/6 = 0$. By Claim 1, the vertex v_1 does not send charge to v_4 and v_2 does not send charge to v_3 .

So we may assume that the two 4-faces are incident. By symmetry, assume that f_1 and f_2 are 3-faces and f_3, f_4 is a 4-face. If v_2 is a crossing vertex, then the final charge of v is $4 - 6 + 3 \times 2/3 = 0$, see Fig (f); if v_2 is a true vertex, then the final charge of v is $4 - 6 + 1 + 2/3 + 1/3 = 0$, see Fig (g). Furthermore, by Claim 1, the vertex v_1 does not send charge to v_2 and v_4 .

Subcase 3.3. Assume that v is incident with one big face having a face angle v_1vv_4 . Firstly, assume that both v_2 and v_3 are true vertices. Then v_1 and v_4 are all crossing vertices since v is not contained in two adjacent triangles in G , see Fig (h). By the discharging rules, the final charge of v is at least $4 - 6 + 3 \times 2/3 = 0$. Moreover, by Claim 1, we know that v_2 does not send charge to v_1 and v_3 does not send charge to v_4 . In a semi-fan, if v_2v is a fan rib or v_3v is a fan rib, then the average charge sent out by the center vertex is $1/3$.

Secondly, assume that one of $\{v_2, v_3\}$, say v_2 , is a crossing vertex. The vertices v_1 and v_3 are all true vertices for crossing vertices are independent, and from the property (P5), v_4 is a crossing vertex. Again by property (P5), the crossing vertex v_2 is incident with two big faces, and v_2 receives 0 from its neighbors. By Claim 1, the crossing vertex v_4 also receives 0 from v_3 . In a semi-fan, if v_3 is the center vertex and v_3v is a fan rib, then the average charge sent out by the center is $1/3$.

(i) If f_1 is a 5^+ -face, then v receives at least $4/3$ from f_1 by Lemma 3, and receives $2/3$ from v_3 , and thus the final charge of v is at least $4 - 6 + 4/3 + 2/3 = 0$.

(ii) If v is incident with a 4-face $f_1 = v_1vv_4v^*$, then v receives 1 from f_1 and receives $1/3$ from v_1 , the final charge of v is $4 - 6 + 1 + 2/3 + 1/3 = 0$, see Fig (i). In this case, the vertex v^* is a big vertex and v_1 sends 0 to v^* . As mentioned above, v_1 also sends 0 to v_2 .

Subcase 3.4. Assume that v is incident with four 3-faces. From property (P5), the vertex v is not contained in adjacent triangles of G , then v is incident with at least two crossing vertices. Consequently, we may assume that v_2 and v_4 are crossing vertices since the crossing vertices are independent, and hence v_1 and v_3 are all true vertices. Moreover, there are two multiple edges of G with ends v_1 and v_3 , a contradiction. Therefore, it is impossible to have four 3-faces incident with v .

Case 4. The vertex v is a crossing vertex.

Clearly, all the neighbors of v are all true vertices.

Subcase 4.1. Assume that v is incident with at least three big faces. In big face f_i , the vertices v_i and v_{i-1} are true vertices, but they are not linked, then one of $\{v_i, v_{i-1}\}$ is a big vertex. So every big face is incident with at least one big vertex, and hence v receives at least $2/3$ from each of its incident big faces. Therefore, the final charge of v is at least $4 - 6 + 3 \times 2/3 = 0$.

Subcase 4.2. Assume that v is incident with two big faces. Firstly, assume that the two 3-faces are adjacent, say f_1 and f_2 . If v_3 is a small vertex, then $\deg_{G^*}(v_2), \deg_{G^*}(v_4) \geq 6$ and $\deg_{G^*}(v_1) \geq \Delta - 2$. The vertex v receives at least $2/3$ from each of its incident big faces and receives $2/3$ from v_1 , and hence the final charge of v is at least $4 - 6 + 3 \times 2/3 = 0$, see Fig (j). In a semi-fan, if v_1 is the center vertex and v_1v is a fan rib, then the average charge sent out by it is $1/3$.

So we may assume that v_3 is a big vertex in G^* . If v_2 and v_4 are all big vertices, then v receives at least 1 from each of its incident big faces, and the final charge of v is at least $4 - 6 + 2 \times 1 = 0$. So, by symmetry, we may assume that v_4 is a small vertex. By (P1), we have $\deg_{G^*}(v_2) \geq \Delta - 2 \geq 9$. Therefore, the vertex v receives at least 1 from f_3 , receives at least $2/3$ from f_4 , receives $1/3$ from v_2 , then its final charge is at least $4 - 6 + 1 + 2/3 + 1/3 = 0$, see Fig (k).

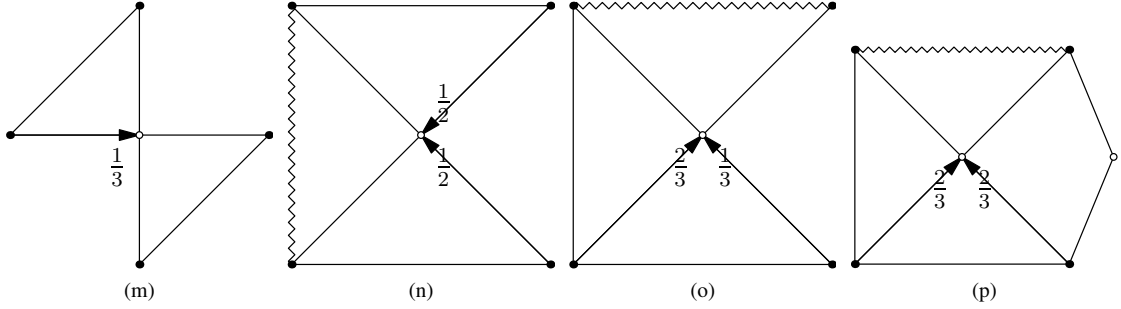
Secondly, assume that the two 3-faces are not adjacent, say f_2, f_4 are two 3-faces. If v_1v_2 is a new edge, then $\deg_{G^*}(v_3), \deg_{G^*}(v_4) \geq \Delta - 2$. By discharging rules, the vertex v receives at least $2/3$ from each of its incident big faces, receives $1/3$ from each of $\{v_3, v_4\}$, then the final charge of v is at least $4 - 6 + 2 \times 2/3 + 2 \times 1/3 = 0$, see Fig (l). Note that v_3 and v_4 send 0 to each other.

By symmetry, we may assume that both v_1v_2 and v_3v_4 are edges of G . If all the vertices of v_1, v_2, v_3, v_4 are big, then v receive at least 1 from each of its incident big faces, and thus the final charge of v is at least $4 - 6 + 2 \times 1 = 0$. So we may assume that there exists a small vertex, say v_4 . Then $\deg_{G^*}(v_2), \deg_{G^*}(v_3) \geq \Delta - 2$ and v_1 is a big vertex. By R2, the vertex v receives at least $2/3$ from f_1 and receives at least 1 from f_3 and receives $1/3$ from v_2 , then its final charge is at least $4 - 6 + 1 + 2/3 + 1/3 = 0$, see Fig (m). Note that v_2 sends 0 to v_1 , so in a semi-fan, if the center sends $1/3$ to such a crossing vertex, then it sends out 0 through its precursor or successor at it.

Subcase 4.3. Assume that v is incident with precisely one big face. Without loss of generality, assume that f_1 is the big face.

(1) Suppose that the edge v_2v_3 is a new edge. From property (P1), $\deg_{G^*}(v_1), \deg_{G^*}(v_4) \geq \Delta - 2$. By the discharging rules, the vertex v receives $1/2$ from each of $\{v_1, v_4\}$ and receives at least 1 from f_1 , and then the final charge of v is at least $4 - 6 + 1 + 2 \times 1/2 = 0$, see Fig (n).

(2) Suppose that one of $\{v_1v_2, v_3v_4\}$ is a new edge. By symmetry, assume that v_1v_2 is a new edge. Then v_3 and v_4 have degree at least $\Delta - 2$. If v receives at least 1 from f_1 , then v receives $2/3$ from v_3 and receives $1/3$ from v_4 , and hence the final charge of v is at least $4 - 6 + 1 + 2/3 + 1/3 = 0$, see Fig (o). So we may assume that v receives less than 1 from f_1 . In fact, the vertex v receives $2/3$ from f_1 , and f_1 is a 4-face with only one big vertex v_4 . So we may assume that $f_1 = v_1vv_4v^*$ and v_1 is a $(3, 4)$ - or $(4, 5)$ -vertex



and v^* is a crossing vertex, see Fig (p). In this case, the vertex v receives $2/3$ from f_1 and receives $2/3$ from each of $\{v_3, v_4\}$, and hence the final charge of v is $4 - 6 + 3 \times 2/3 = 0$.

(3) All edges v_1v_2, v_2v_3, v_3v_4 are edges of G . From Lemma 1, the vertices v_1 and v_4 are not adjacent in G , and then v_1, v_2, v_3, v_4 induced a K_4^- in G . Graph G satisfies the property \mathcal{P} and (P4), so one of $\{v_2, v_3\}$, say v_2 , is a vertex of degree at most five in G by the hypothesis. Then each of v_1, v_3, v_4 has degree at least $\Delta - 2$. The vertex v receives at least 1 from f_1 and receives $1/3$ from each of v_1, v_3, v_4 , see Fig (q). Therefore, the final charge of v is at least $4 - 6 + 1 + 3 \times 1/3 = 0$.

Subcase 4.4. Suppose that v is incident with four 3-faces. From Lemma 1, there exists a new edge between v_1, v_2, v_3, v_4 . Assume that v_1v_2 is a new edge, then $\deg_G(v_3), \deg_G(v_4)$ are of degree at least $\Delta - 2$, and v_3v_4 is contained in two triangles $v_1v_3v_4v_1$ and $v_2v_3v_4v_2$ of G , which contradicts the hypothesis.

Case 5. The vertex v is a $(4, 5)$ -vertex or $(5, 5)$ -vertex.

Subcase 5.1. If v is incident with at least two big faces, then v receives at least $1/2$ from each of its incident big faces, and hence the final charge is at least $5 - 6 + 2 \times 1/2 = 0$.

Subcase 5.2. Assume that v is incident with precisely one big face, say f_4 . If v receives at least 1 from f_4 , then the final charge of v is at least $5 - 6 + 1 = 0$. So we may assume that v receives less than 1 from f_4 .

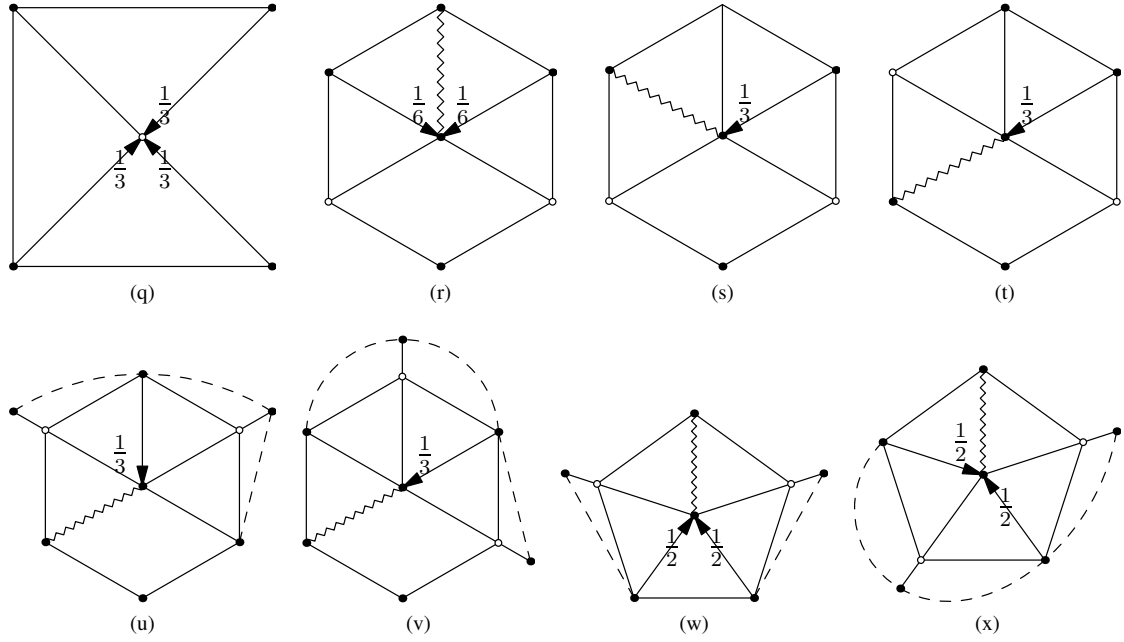
Firstly, assume that neither vv_3 nor vv_4 is a new edge. By Lemma 3, the face f_4 is a 4-face and v_3, v_4 are all crossing vertices, the vertex v receives $2/3$ from f_4 . Moreover, v_2 and v_5 are all true vertices. If there is no new edge at v , i.e., v is a $(5, 5)$ -vertex, then v receives $1/6$ from v_2 and v_5 , its final charge is $5 - 6 + 2/3 + 2 \times 1/6 = 0$. If v_1v is a new edge, then v receives $1/6$ from each of $\{v_2, v_5\}$, the final charge of v is $5 - 6 + 2/3 + 2 \times 1/6 = 0$, see Fig (r). By symmetry, if one of $\{vv_2, vv_5\}$, say vv_2 , is a new edge, then v receives $1/3$ from v_5 , and the final charge of v is $5 - 6 + 2/3 + 1/3 = 0$, see Fig (s); note that v_1 is either a true or a crossing vertex.

Secondly, assume that one of vv_3, vv_4 is a new edge, by symmetry, we may assume that vv_3 is a new edge. If both v_1 and v_5 are true vertices, then the local structure is as illustrated in Fig (t), the final charge of v is $5 - 6 + 2/3 + 1/3 = 0$. If v_1 is a true vertex and v_5 is a crossing vertex, then the local structure is as illustrated in Fig (u), v receives $1/3$ from v_1 , the final charge of v is $5 - 6 + 2/3 + 1/3 = 0$. If v_1 is a crossing vertex and v_5 is a true vertex, then the local structure is as illustrated in Fig (v), v receive $1/3$ from v_5 , the final charge of v is $5 - 6 + 2/3 + 1/3 = 0$.

Subcase 5.3. The vertex v is incident with five 3-faces.

(1) If v is a $(5, 5)$ -vertex, then at least three of its neighbors in G^* are true vertices, the vertex v receives $1/3$ from each of these vertices. The final charge of v is at least $5 - 6 + 3 \times 1/3 = 0$.

(2) Assume that v is a $(4, 5)$ -vertex and vv_1 is a new edge.



If both v_3 and v_4 are true vertices, then v_2 and v_5 are all crossing vertices by property (P5), the vertex v receives $1/2$ from each of v_3, v_4 , and its final charge is $5 - 6 + 2 \times 1/2 = 0$, see Fig (w).

If one of $\{v_3, v_4\}$, by symmetry, say v_3 , is a crossing vertex, then v_2, v_4 are all true vertices and v_5 is a crossing vertex. By discharging rules, the vertex v receives $1/2$ from each of v_2, v_4 . Therefore, the final charge of v is $5 - 6 + 2 \times 1/2 = 0$, see Fig (x).

If v is a $(3, 5)$ -vertex, then the final charge is $5 - 6 + 1 = 0$ by R1.

If $\deg_{G^*}(v) = 6$, then the final charge of v is nonnegative.

If the vertex v has degree $7, \dots, \Delta - 3$ in G^* , then its final charge is positive.

The remainder case is v of degree at least $\Delta - 2 \geq 9$.

In a semi-fan, assume that the center vertex sends out a $2/3$ through a fan rib. If the center vertex sends a $2/3$ to a small vertex as illustrated in Fig (b) (f) (h) (i) (j), then we have proved that the average charge sent out by the center is $1/3$.

Suppose that the center is the vertex v_3 in Fig (o) or (p). If v_3 sends 0 to v_2 , then the average charge sent out by v_3 is $1/3$. If v_3 sends a positive charge to v_2 , then v_2 is a $(3, 4)$ - or $(4, 5)$ -vertex. But v_2 can not be a $(3, 4)$ -vertex, otherwise v_2 is contained in a triangle $v_2v_3v_4v_2$ of G , which contradicts (P4). By the local structure of v_2, v and v_3 , if v_3 sends a positive charge to v_2 , then the vertex v_2 is a $(4, 5)$ -vertex and this $(4, 5)$ -vertex receives $1/3$ from the center (see Fig (s)), v_3 sends out 0 through its precursor or successor, hence the average charge sent out by the center is $(2/3 + 1/3)/3 = 1/3$.

Suppose that the center is the vertex v_4 as illustrated in Fig (p). If v_4 sends 0 to v^* , then the average charge sent out by v_4 is $1/3$. On the other hand, by the local structure around v and v^* , it is impossible that v_4 sends a positive charge to v^* .

In what follows, we assume that the center does not send a $2/3$ through fan ribs.

Assume that the center sends out a $1/2$ through a fan rib. Suppose that the center sends a $1/2$ to a crossing vertex. By symmetry, we may assume that the center is the vertex v_1 as illustrated in Fig (n). If v_1 sends a positive charge to v_2 , then v_2 is a $(3, 4)$ - or $(4, 5)$ -vertex. If v_2 is a $(3, 4)$ -vertex, then v_1 sends $1/3$ to v_2 and it sends 0 through the precursor or successor. If v_2 is a $(4, 5)$ -vertex and it is the vertex v as

illustrated in Fig (s), then the center sends $1/3$ to such a $(4, 5)$ -vertex and sends 0 through the precursor or successor. If v_2 is a $(4, 5)$ -vertex and it is the vertex v as illustrated in Fig (w) or Fig (x), then the center sends $1/2$ to such a $(4, 5)$ -vertex, but it sends 0 to through the precursor or successor. By the above arguments, the average charge sent out by the center is at most $(4 \times 1/2)/5 = 2/5$, the equality holds if and only if the semi-fan contains five faces and the center sends out four $1/2$.

Assume that the center sends out a $1/2$ to a $(4, 5)$ -vertex, but not to crossing vertices. By the discharging rules, the center sends $1/2$ “near” the big fan ribs, so the average charge sent out by the center is at most $(2 \times 1/2 + (k - 3) \times 1/3)/k = 1/3$.

If the center sends out at most $1/3$ through each fan ribs, then the average charge sent out by the center is less than $1/3$.

Therefore, if v is a $(\Delta - 2)$ -vertex, then it only sends positive charge to crossing vertices or $(5, 5)$ -vertices, then the average charge sent out by the centre is at most $1/3$, the final charge of v is at least $\Delta - 2 - 6 - (\Delta - 2) \times 1/3 \geq 0$; the equality holds if and only if $\Delta = 11$ and the average charge sent out by center in every semi-fan is exactly $1/3$.

If v is a $(\Delta - 1)$ -vertex, then it only sends positive charge to crossing vertices or $(4, 4)$ or $(4, 5)$ - or $(5, 5)$ -vertices, then the average charge sent out by the centre is at most $2/5$, the final charge of v is at least $(\Delta - 1) - 6 - (\Delta - 1) \times 2/5 \geq 0$; the equality holds if and only if $\Delta = 11$ and the average charge sent out by the center is $2/5$.

If v is a Δ -vertex and it is not adjacent to any 3-vertices of G , then the final charge of v is at least $\Delta - 6 - \Delta \times 2/5 > 0$; if v is a Δ -vertex and v is adjacent to a 3-vertex of G , then its final charge is at least $\Delta - 6 - 1/2 - \Delta \times 2/5 > 0$.

Let w be a vertex of G^* with maximum degree.

If $\deg_{G^*}(w) = \Delta$, then the final charge of w is positive.

If $\deg_{G^*}(w) = \Delta - 1$, then it can not send charge to the $(4, 5)$ -vertex as illustrated in Fig (w) or (x) by (P3). Therefore, the average charge sent out by w is less than $2/5$, and hence the final charge of w is positive.

If $\deg_{G^*}(w) = \Delta - 2$, then it can not send charge to the $(5, 5)$ -vertex which is incident with five 3-faces. Therefore, w can only send charge to crossing vertices, and hence the final charge of w is positive.

If $\deg_{G^*}(w) \in \{7, \dots, \Delta - 3\}$, then the final charge of w is $\deg_{G^*}(w) - 6 > 0$.

By the hypothesis that $\Delta \geq 11$ and (P1), there exists at least one vertex having degree at least 7.

Therefore, the final charge of vertices with maximum degree are positive and the sum of the final charge of each elements is positive, which derive a contradiction. This complete the proof the theorem.

Remark 1. Zhang et al. [13] proved that TCC holds for 1-planar graphs with maximum degree at least 13; and we also can extend this result to 1-toroidal graphs with maximum degree at least 13 using similar techniques in this paper.

Problem 1. Whether this method can be used to prove similar result for the diamond-free 1-toroidal graphs?

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